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A class of exact pp-wave string models with interacting light-cone gauge actions

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Abstract: We find a general class of pp-wave string solutions with NS-NS H_3 or R-R F_3 field strengths, which are analogous to solutions with non-constant F_5 recently considered by Maldacena and Maoz (hep-th/0207284). We show that: (i) all pp-wave solutions supported by non-constant H_3 or F_p fields are exact type II superstring solutions to all orders in α'; (ii) the corresponding light-cone gauge Green-Schwarz actions are non-linear in bosons but always quadratic in fermions, and describe UV finite 2-d theories; (iii) the pp-wave backgrounds supported by non-constant F_3 field do not have, in contrast to their F_5-field counterparts, "supernumerary" supersymmetries and thus the associated light-cone GS actions do not possess 2-d supersymmetry. We consider a specific example where the pp-wave F_3 background is parametrized by an arbitrary holomorphic function of one complex bosonic coordinate. The corresponding GS action has the same bosonic part, similar Yukawa terms but twice as many interacting world-sheet fermions as the (2,2) supersymmetric model originating from the analogous F_5 background. We also discuss the structure of massless scalar vertex operators in the models related to $N = 2$ super sine-Gordon and $N = 2$ super Liouville theories.

Keywords: Penrose limit and pp-wave background, Superstrings and Heterotic Strings, Superstring Vacua.

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1. Introduction

Finding new string models with Minkowski signature which are exact in α' and whose spectrum can be explicitly determined is of great interest from the point of view of better understanding of string theory in curved (cosmological, black-hole, etc.) backgrounds.

One class of such models has metric admitting a covariantly-constant null Killing vector. Many of such pp-wave backgrounds with metric and other fields having “null” structure [1, 2] are simple examples of α'-exact solutions of string thesallory (see [3] for a review).

Some of these backgrounds give exactly solvable (in terms of free oscillators in light-cone gauge) non-compact curved-space superstring models, for which one can find the string spectrum and compute some simplest "observables" (partition function, some correlation functions, etc.) in much the same way as in flat space [4]. As was recently realized [5]–[8], this solvability property applies to string models corresponding not only to the NS-NS but also to certain R-R [9] plane-wave backgrounds.

It is then natural to look for more complicated cases where the light-cone action is no longer quadratic but may be integrable, so that the corresponding string spectrum may still be possible to determine. An interesting example of such models (representing a pp-wave metric supported by a particular non-constant R-R 5-form background) was recently proposed in [10].

In this paper we first study a general class of NS-NS models based on pp-wave metric and "null" 3-form H_3 background depending on arbitrary harmonic functions $b_m(x)$ ($\partial^2 b_m = 0,\ m = 1, 2, \ldots$) of the transverse coordinates x_i. The special case when b_m are linear in x_i corresponds to homogeneous plane-wave backgrounds with constant H_3 field (these are, in fact, WZW models [11]). Another special case which we consider is where b_m are chosen to be holomorphic functions of complex combinations ($z_1 = x_1 + ix_2, \ldots$) of coordinates x_i. The R-R counterpart of this background — with H_3 replaced by the R-R 3-form F_3 — is the direct analog of the F_5-solution of [10].

We shall demonstrate that all such backgrounds (both the NS-NS and the R-R ones, including the one of [10]) are exact superstring solutions to all orders in α': all corrections to the leading-order field equations should vanish in a natural scheme. A direct world-sheet proof of exactness of the non-constant F_5-background of [10] (and certain other supersymmetric pp-wave backgrounds) was recently given in [12]. The space-time argument presented here is universal: it applies to general pp-wave string models with a non-constant F_p field.

The NS-NS models describe a new class of exact string solutions, different from the chiral null models [2] which are also exact in α' but in which the form of the B_2 background is correlated with the off-diagonal terms in the metric.

For the pp-wave background with a R-R F_3 field, the holomorphic functions parametrizing the background can be chosen so that the bosonic part of the light-cone gauge GS lagrangian is that of an integrable (e.g., super sine-Gordon or super Liouville) model. The same pp-wave metric can be supported by different R-R field strengths, leading to solutions with different amounts (and types of) of supersymmetry. In contrast to the F_5 models of [10], here the fermionic GS extension of the same bosonic light-cone action, while still representing a UV finite 2-d theory, is not (2,2) supersymmetric: in the F_3 case there are twice as many fermions and the coefficient of the Yukawa interaction term is smaller by factor of $\sqrt{2}$. The corresponding string background preserves 8 space-time supersymmetries, whose role in the light-cone gauge is to imply the existence of the same number of massless fermions which are decoupled from the rest of the fields.

There are several reasons which make these non-supersymmetric 1+1 dimensional models worth of further study. In particular, like their (2,2) supersymmetric counterparts, they may turn out to be integrable. This property would allow one to determine the correspond-

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ing string spectra. Another interesting question is whether the NS-NS models describing similar inhomogeneos pp-wave backgrounds, which have an advantage of their covariant-gauge action (having as usual (1,1) world-sheet supersymmetry) being explicitly known, may also be integrable.

Before proceeding to the main topic of this paper, let us recall some other previously known embeddings of interacting non-conformal 2-d theories (with d-dimensional euclidean-signature target space) into α'-exact conformal sigma models (with $2+d$ dimensional Minkowski-signature target space).

Given a generic non-conformal sigma model with a curved euclidean d-dimensional space $ds^2_d = g_{ij}(x)dx^i dx^j$ as a target space and with the RG beta-function $\beta_{ij}(g) = R_{ij} + \cdots$, one can construct [18] a special Weyl-invariant sigma-model (i.e. a string solution) with the following $2+d$ dimensional Minkowski-signature metric and the dilaton

$$ds^2_{2+d} = dudv + g_{ij}(x,u)dx^i dx^j \,, \qquad \phi(x,u,v) = v + \tilde{\phi}(u,x) \,. \tag{1.1}$$

Here $g_{ij}(x,u)$ is subject to the 1-st order differential equation

$$\frac{\partial}{\partial u} g_{ij}(x,u) = \beta_{ij}(g(x,u)) \,, \tag{1.2}$$

which is nothing but the RG equation in the "transverse" d-dimensional theory, with u interpreted as a logarithm of the 2-d UV scale. A class of exact string solutions [18] is found, in particular, in the case when the transverse model is a (2,2) supersymmetric Einstein-Kahler sigma model (for which the β_{ij}-function has only the one-loop term). The simplest $2+d=4$ dimensional example is provided by the $O(3)$ sigma model with $g_{ij}(x)$ being the S^2-metric. Here $g_{ij}(x,u) = ug_{ij}$ and thus

$$ds^2_{2+d} = dudv + u(d\theta^2 + \sin^2\theta d\varphi^2) \,, \qquad \phi(x,u,v) = v + \frac{1}{4}\ln u \,. \tag{1.3}$$

The key feature of these solutions is that the dilaton is non-trivial, and contains, in particular, the term linear in v. This means that in the light-cone gauge description, the non-conformal transverse theory should be supplemented by the definition of the stress tensor following from the original "$2+d$-dimensional" covariant action.

In another example we would like to mention one promotes an interacting Toda-type 2-d QFT to a Minkowski-signature string solution in $2+d$ dimensions by using the following pp-wave background with linear dilaton [20] (ρ_i=const, $i = 1, \ldots, d$)

$$ds^2_{2+d} = dudv + K(x)du^2 + dx_i dx_i \,, \qquad \phi = \rho_i x_i \,. \tag{1.4}$$

The string equations are then satisfied to all orders provided

$$\partial_i\partial_i K - 2\rho_i\partial_i K = 0 \,, \qquad \rho_i\rho_i = \frac{1}{4\alpha'}(8-d) \,. \tag{1.5}$$

The solution for K can be chosen as a sum of exponents: $K = \sum_n c_n e^{\alpha_{ni}x_i}$, where for each n we should have $\alpha_{ni}\alpha_{ni} = 2\rho_i\alpha_{ni}$. A particular case is provided by the Toda model

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potential, i.e. we end up with the Toda model as the light-cone gauge theory.[1] The sigma model corresponding to (1.4) is T-dual (in $y = u - v$ coordinate) to a sigma model associated with a particular G/H "null-gauged" WZW model [20].[2]

The rest of the paper is organized as follows.

In section 2 we present a class of pp-wave solutions with non-constant H_3 form and constant dilaton. We argue that all similar pp-wave backgrounds supported by NS-NS or R-R fields (including the ones of [10]) should be exact solutions of superstring theory. We also determine conditions for residual supersymmetry of the H_3-background. In section 3 we discuss a subclass of NS-NS solutions which are parametrized by holomorphic functions, determining the fractions of supersymmetry they preserve. In section 4 we write down the (1,1) supersymmetric RNS sigma model lagrangians for these backgrounds.

In section 5.1 we find the light-cone gauge GS actions associated with pp-wave backgrounds with non-constant F_p fields. We first show that the corresponding interacting 2-d actions are always quadratic in GS fermions and thus are easy to write down. We then prove (in section 5.2) that these light-cone actions define UV finite 2-d theories, in agreement with the general α'-exactness argument of section 2.2.

In section 5.3 we consider explicitly the light-cone GS action for the pp-wave R-R F_3 background which is "S-dual" to the NS-NS background of section 2.1. This background does not admit "supernumerary" Killing spinors (i.e. the ones which are not annihilated by the light-cone gauge condition [32]), and thus the associated light-cone GS action does not have "accidental" linearly realized 2-d supersymmetry. This is in contrast to the case of F_5-background of [10] where a similar model (with the same bosonic part) is (2,2) supersymmetric.

Finally, in section 6, we consider two examples of backgrounds parametrized by a holomorphic function, which are related to integrable models. We present a discussion of the structure of the associated massless vertex operators, which applies also to the model of [10].

2. Supersymmetric pp-wave model with non-constant NS-NS 3-form background

2.1 The form of the string solution

Let us consider the following ansatz for the metric and NS-NS 2-form potential in 10-dimensional superstring theory

$$ds^2 = dudv + K(x)du^2 + dx_i^2 + dy_m^2 \,, \qquad i = 1, \ldots, d \,, \qquad m = d+1, \ldots, 8 \,, \quad (2.1)$$

$$B_2 = b_m(x) du \wedge dy_m \,, \qquad H_3 = \partial_i b_m(x) dx_i \wedge du \wedge dy_m \,. \quad (2.2)$$

[1] Explicitly, the Toda model corresponds to the case when α_i are simple roots of the Lie algebra of a maximally non-compact real Lie group G of rank $N = d - 2$ and $\rho_i = (1/2)\sum_n \alpha_{ni}$ is half of the sum of all positive roots.

[2] There H is a nilpotent subgroup of G generated by $N-1$ simple roots (this condition on H is needed to get a model with only one time-like direction). The flat transverse coordinates x^i correspond to the Cartan subalgebra generators.

We split the 8 transverse coordinates into two groups — x_i and y_m, with the functions K and b_m depending only on x_i. We set the dilaton to be constant, but there is a straightforward generalization to the case when $\phi = \rho_i x_i + \tilde{\phi}(u)$ (which makes possible to include the background (1.4), (1.5) as a special case). Another obvious generalization is to allow K and b_m to depend on u.

In view of the structure of the background (in particular, the fact that the non-zero part of curvature of the metric (2.1) is $R_{uiuj} = -(1/2)\partial_i\partial_j K$) the non-trivial components of the leading-order string field equations are $D_i H_{iku} = 0$ and $R_{uu} - \frac{1}{4}H_{ukl}H_u{}^{kl} = 0$. They imply (we always assume summation over i and m)[3]

$$\partial_i\partial_i b_m = 0\,, \qquad \partial_i\partial_i K + \partial_i b_m \partial_i b_m = 0\,. \tag{2.3}$$

Thus b_m can be any set of harmonic functions of x_i, while the general solution for K can be written as

$$K = -\frac{1}{2}b_m b_m + K_0\,, \qquad \partial_i\partial_i K_0 = 0\,. \tag{2.4}$$

There are several special cases. For $b_m = 0$ we recover the standard pp-wave solution with $K = K_0(x)$ being a harmonic function. For linear b_m, i.e. constant H_3, and $K_0 = 0$ we get

$$b_m = f_{mi}x_i\,, \qquad H_3 = f_{mi}dx_i \wedge du \wedge dy_m\,, \qquad K = -\frac{1}{2}w_{ij}x_i x_j\,, \qquad w_{ij} \equiv f_{mi}f_{mj}\,, \tag{2.5}$$

where w_{ij} (which is the mass matrix for x_i in the light-cone gauge action) is non-negative. The corresponding models can be interpreted as WZW theories for non-semisimple groups nw (see also [4, 8]).[4]

Another regular solution of the equation for b_m which does not require an introduction of singularities or sources is given by a quadratic "traceless" form:[5]

$$b_m = c_{mij}x_i x_j\,, \qquad c_{mii} = 0\,, \qquad K = -\frac{1}{2}d_{ijkl}x_i x_j x_k x_l\,, \qquad d_{ijkl} \equiv c_{mij}c_{mkl}\,. \tag{2.6}$$

This gives a quartic non-negative potential in the light-cone gauge.[6] An example of a singular solution (which needs to be supported by a delta-function source) is $b_m = Q_m/x^{d-2}$ and $K = -Q_m^2/(2x^{2d-4})$.[7]

The Laplace equation for b_m in (2.3) can be solved also by choosing b_m to be, e.g., real part of holomorphic functions of complex combinations of coordinates $z_1 = x_1 + ix_2$, etc. The corresponding string models parametrized by holomorphic functions will be discussed below in section 3. Their R-R counterparts (with H_3 replaced by "S-dual" F_3

[3] In the case of a non-trivial dilaton $\phi = \rho_i x_i$ these equations become (cf. (1.5)): $(\partial_i - 2\rho_i)\partial_i b_m = 0$, $(\partial_i - 2\rho_i)\partial_i K + \partial_i b_m \partial_i b_m = 0$.

[4] In this case the generalized curvature $\hat{R}$ discussed in the next subsection 2.2 vanishes, as it should be for a parallelizable space.

[5] In what follows we set the solution K_0 of the homogeneous equation in (2.4) to zero.

[6] The potential will always have flat directions. For example, one can choose $b_m = c_m(x_1^2 - x_2^2)$, so that $K = -(1/2)c^2(x_1^2 - x_2^2)^2$.

[7] Note that such K decays twice as fast as the harmonic function $K = K_0$ of the standard pp-wave solution with $H_3 = 0$.

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background) are direct analogs of the pp-wave solution [10] supported by a non-constant 5-form background. Such solutions will, in general, be singular at some points, i.e. they will require extra assumptions about sources or string-theory resolution of the singularities (cf. [19]).

Finally, let us mention that lifts of the above solutions to 11 dimensions belong to a class of $D = 11$ pp-wave backgrounds first considered in [16] (a general discussion of pp-waves in $D = 10$ supergravity appeared also in [17]). There the 4-form background was chosen as $F_4 = f_3 \wedge du$ with 3-form f_3 being any closed and co-closed form and K-function in the metric satisfying $\partial^2 K = |f_3|^2$. The present NS-NS solutions correspond to a specific choice of $f_3 = \partial_i b_m(x)\ dx_i \wedge dy_m \wedge dx_{11}$. The focus of the present paper is not on studying general types of pp-wave solutions of supergravity as such, but on identifying backgrounds which are exact in α', and which lead to new exactly solvable superstring models.

2.2 Exactness in α'

We shall now show that the above background represents an exact string solution, i.e. there exists a scheme in which it is not modified to all orders in α'. We shall first give a general argument based on the structure of low-energy effective action in closed superstring theory. This argument will apply not only to the NS-NS background (2.1), (2.2) but also to any similar R-R background with the pp-wave metric (2.1) supported by a p-form field strength which has a "null" form

$$F_p = f_{k_1 \ldots k_{p-1}}(x) dx^{k_1} \wedge \cdots \wedge dx^{k_{p-1}} \wedge du\,. \tag{2.7}$$

In particular, this will also demonstrate the exactness of the pp-wave background considered in [10]. We shall then present a different argument (based on certain very plausible conjecture on the structure of the beta-functions in the general bosonic or (1,1) supersymmetric sigma model) which will independently suggest the exactness of the NS-NS background (2.1), (2.2) already in the bosonic string theory.

Let us start with a digression on the structure of the type II superstring effective action. By definition, the effective action for the massless string modes is constructed so that to reproduce the string S-matrix. Field redefinitions [21, 22, 15, 14] allow one to avoid "quadratic" or "propagator correction" terms (i.e. terms whose weak-field expansion starts with quadratic terms). In addition, as is well known [23], the on-shell superstring amplitudes for massless modes do not contain (in contrast to the bosonic string amplitudes) α'-corrections, i.e. the supergravity 3-point amplitudes are exact. This suggests that there may exist a field-redefinition choice (or, in the β-function context, a 2-d RG scheme) in which the weak-field expansion of the α'-dependent part of the effective action starts with quartic terms only (i.e. $\alpha'^3 RRRR + \cdots$). Indeed, since the "quadratic" and "cubic" terms that may be present in the action should not contribute to the S-matrix, it may be possible to redefine them away. More precisely, for our present purpose, it is sufficient to argue that in such a scheme the effective lagranigian will not contain terms like

$$\Delta L = a_1(D \ldots DR)^2 + a_2(D \ldots DF)^2 + a_3\ R_{\mu.\nu.} D \ldots DF_{\mu \ldots}\ D \ldots DF_{\nu \ldots}\,. \tag{2.8}$$

Here R is curvature, D is covariant derivative and $F = dC$ stands for any p-form field strength of the type II supergravity multiplet, including H_3. In this case there will be no terms like

$$c_1 D \ldots D R_{\mu.\nu.} + c_2 D \ldots D F_{\mu\ldots} D \ldots D F_{\nu\ldots} \tag{2.9}$$

in the equation for the metric, and the terms like $D \ldots D F$ in the equation for the p-form field.

To prove that the terms in (2.8) can either be redefined away or modify the 3-point S-matrix (and thus are excluded in the superstring case), let us make several observations.[8] We shall concentrate on the a_3-terms in (2.8) since the argument for a_1- and a_2- terms is trivial. First, as we will be interested only in the c_2-terms in (2.9) which do not contain extra powers of R, we may ignore commutators of covariant derivatives in (2.8) (it is, in fact, sufficient to replace D_μ in (2.8) by ∂_μ). Second, the a_3-terms in (2.8) which contain (after integration by parts) two contracted derivatives acting on one field, i.e. $D^2 R$ or $D^2 F$ factors, can be eliminated by a local metric or potential C redefinition in the standard $R + F^2$ kinetic term. Indeed, using Bianchi identities and integrating by parts, such terms can be put into the form proportional (up to higher-order terms that can be ignored) to the leading parts of equations of motion, i.e. they can be written as $R_{\mu\nu} X_{\mu\nu}$ and $D_\mu F_{\mu\nu\ldots} Y_{\nu\ldots}$, and thus can be redefined away. Next, the only a_3-terms in (2.8) that may contribute to the on-shell 3-point amplitudes must have all indices of D-derivatives contracted with field (polarization tensor) indices and not between themselves: for massless on-shell amplitudes $p_1 + p_2 + p_3 = 0$, $p_i^2 = 0$, $i = 1, 2, 3$, and thus $p_i \cdot p_j = 0$. Let us consider, for concreteness, the case of $F = F_3$. Then the a_3-terms that may contribute to the S-matrix can only have at most two D-derivative factors (the total number of indices of $h_{\mu\nu}$ and $C_{\mu\nu}$ in the 3-point vertex should be greater or equal to the total number of derivatives). Explicitly, such terms are $R_{\mu\alpha\nu\beta} F_{\mu\alpha\rho} F_{\nu\beta\rho}$ and $R_{\mu\alpha\nu\beta} D_\rho F_{\mu\alpha\sigma} D_\sigma F_{\nu\beta\rho}$ (other contractions of derivatives are equivalent or give vanishing contribution to the 3-point amplitudes). These terms can not be present in the superstring effective action.

Finally, let us show that all other possible terms with contracted derivatives that do not contribute to the S-matrix can indeed be redefined away. Consider the generic case of two contracted derivatives (as already mentioned, positions of derivatives in $D \ldots D$ factors in (2.8) are not important): $L_3 = R_{\mu.\nu.} D \ldots D D_\lambda F_{\mu\ldots} D \ldots D D_\lambda F_{\nu\ldots}$. Since terms with $D^2 F_{\mu..}$ can be redefined away, we can write $L_3 = R_{\mu.\nu.} D^2 (D \ldots D F_{\mu..} D \ldots D F_{\nu..})$, or, integrating by parts, as $L_3 = -D^2 R_{\mu.\nu.} D \ldots D F_{\mu..} D \ldots D F_{\nu..}$. Since $D^2 R_{\mu\beta\nu\rho} \to D_\mu D_\nu R_{\beta\rho} + \cdots$, such terms can be again redefined away.

Returning to our main argument, let us now take into account the specific form of the background in question (2.1) and (2.7). First, we note that its "null" structure implies that all scalar invariants constructed out of R, D and F identically vanish. The only non-zero components of R and F have, respectively, two and one of u-indices, and the only non-zero component of the covariant derivative is ∂_i (the fields do not depend on v). This implies that the only non-trivial correction to the equation $DF + \cdots = 0$ for F must be linear in F, i.e. it can only have the structure $D \ldots D F_{u\ldots}$. Such terms that could

[8] We are grateful to R. Metsaev for an important criticism that helped to clarify the argument below.

only originate from the a_2 term in the effective action (2.8) are prohibited by the above argument. Similarly, any non-zero correction to the Einstein equation must carry two u-indices. Since the curvature has only R_{uiuj} components, the required 2-nd rank tensor that can be constructed from such curvature is R_{uu} or $D \ldots DR_{u.u.}$. Thus corrections to the Einstein equation (or to the β-function for the metric) could only be of the form

$$d_1 D \ldots DR_{u.u.} + d_2 D \ldots DF_{u\ldots} D \ldots DF_{u\ldots} . \tag{2.10}$$

They could only follow from the terms (2.9) in the covariant expression (note the position of indices on F-factors), which, however, should be absent, as explained above, in the natural scheme in which the 2- and 3-point terms in the effective action are not modified from their supergravity values. We conclude that in this scheme the background (2.1), (2.7) or (2.2) that solves the leading-order supergravity equations of motion remains the solution to all orders in α'-expansion.

Let us now present an alternative argument that will apply only to the NS-NS case but will be valid also in the bosonic string case: it will suggest that the NS-NS sigma model corresponding to (2.1), (2.2) is conformal to all orders in α'. It is based on a plausible conjecture [15] that for the general sigma model $L = (G_{\mu\nu} + B_{\mu\nu})(x)\partial x^\mu \partial x^\nu$ there should exist an RG scheme in which all α' corrections to the beta-functions for $G_{\mu\nu}$ and $B_{\mu\nu}$ are at least *linear* in the generalized curvature $\hat{R}_{+\mu\nu\lambda\rho} (= \hat{R}_{-\lambda\rho\mu\nu})$ defined by the connection $\hat{\Gamma}^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} \pm (1/2) H^\lambda{}_{\mu\nu}$. This conjecture is supported, in particular, by the explicit higher-loop computations [24], and by the fact that parallelizable spaces (corresponding to WZW models) are independently known to be finite [25]. In such a scheme all corrections to the β-functions for $G_{\mu\nu}$ and $B_{\mu\nu}$ must be of the form

$$\beta_{\mu\nu} = P^{\kappa\lambda\rho\sigma}_{\mu\nu}(R, H, D) \hat{R}_{+\kappa\lambda\rho\sigma} . \tag{2.11}$$

In the present case of (2.1), (2.2) one finds that the only non-zero component of the generalized curvature is

$$\hat{R}_{+uiuj} = -\frac{1}{2}\partial_i\partial_j K - \frac{1}{4}\partial_i b_m \partial_j b_m , \qquad \hat{R}_{+unum} = -\frac{1}{4}\partial_i b_n \partial_i b_m , \qquad \hat{R}_{+jmiu} = \partial_i \partial_j b_m . \tag{2.12}$$

Note that $\hat{R}_{\mu\nu} = 0$ on the equations of motion (2.3). Since the functions K and b_m do not depend depend on v, and any corrections are possible only to β_{uu}. For the present background D_u, D_m are trivial and H_3 and $\hat{R}$ have at least one u-index. This implies that possible corrections must have the structure $\beta_{uu} = O_{ij} \hat{R}_{+uiuj}$, where O_{ij} is a differential operator involving D_i only. Since here $D_i = \partial_i$, we get $O_{ij} = k_1 \partial_i \partial_j + k_2 \partial_i \partial_j \partial^2 + \cdots$ (dots represent terms with higher powers of ∂^2). As was noted above, such "propagator-correction" terms can be in general redefined away (they should, in fact, be absent in the minimal subtraction scheme).

2.3 Conditions for space-time supersymmetry

Let us now determine the conditions under which the background (2.1), (2.2), (2.3) preserves a fraction of type II space-time supersymmetry. Explicit examples will be given later in section 3.

In the case of the metric and H_3 given by (2.1), (2.2) and constant dilaton the type-IIB dilatino transformation law gives the condition

$$\partial_i b_m \Gamma^u \Gamma^{im} \epsilon = 0\,. \tag{2.13}$$

The condition of the vanishing of the gravitino variation is[9]

$$\left[\partial_\mu + \frac{1}{4}\left(\omega_{\hat\lambda\hat\rho\mu} \pm \frac{1}{2} H_{\hat\lambda\hat\rho\mu}\right)\Gamma^{\hat\lambda\hat\rho}\right]\epsilon = 0\,. \tag{2.14}$$

The v-component is solved provided $\partial_v \epsilon = 0$, i.e. $\epsilon = \epsilon(u, x, y)$. The i and m components give[10]

$$\partial_i \epsilon + \frac{1}{4}\partial_i b_m \Gamma^{um}\epsilon = 0\,, \tag{2.15}$$

$$\partial_m \epsilon - \frac{1}{4}\partial_i b_m \Gamma^{ui}\epsilon = 0\,. \tag{2.16}$$

For non-constant H_3, i.e. $\partial_i \partial_j b_m \neq 0$, these equations imply[11]

$$\Gamma^u \epsilon = 0\,. \tag{2.17}$$

As a result, ϵ is independent of x^i and y^m. In general, the u-component of the gravitino variation gives

$$\partial_u \epsilon + \frac{1}{4}\partial_i K \Gamma^{ui}\epsilon - \frac{1}{4}\partial_i b_m \Gamma^{im}\epsilon = 0\,. \tag{2.18}$$

In view of eq. (2.17) and the fact that ϵ is independent of x while $\partial_i b_m$ is a function of x, we get the condition

$$\partial_i b_m \Gamma^{im}\epsilon = 0\,, \tag{2.19}$$

as well as $\partial_u \epsilon = 0$.

The condition (2.17) (or (2.19)) ensures the vanishing of the dilatino variation (2.13). The number of remaining supersymmetries thus depends on existence of constant ϵ solutions of (2.19).

One important conclusion is that the pp-wave background (2.1), (2.2) with non-constant H_3 (or its R-R counterpart with $H_3 \to F_3$ discussed below) does not have "super-numerary" supersymmetries for which $\Gamma^u \epsilon \neq 0$ (i.e. solutions for ϵ which are "orthogonal" to the ones satisfying the GS light-cone gauge condition (2.17)). As a result, in contrast to the case considered in [10], we should not expect to find extra linearly realized supersymmetries in the corresponding light-cone GS action [6, 32, 10, 33]. Still, the light-cone actions we shall get will have several features in common with the one corresponding to the F_5 case discussed in [10].

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[9] Below the sign of H_3-term is chosen as plus; the case of the minus sign (corresponding to $b_m \to -b_m$) leads to equivalent conclusions.

[10] We use that $H_{ium} = \partial_i b_m$ and the tangent-space components of the Lorentz connection given in eq. (4.4) below. We also omit hats on indices of Γ-matrices as the corresponding vierbein components are trivial.

[11] This is the integrability condition for (2.15), (2.16) which is found by multiplying (2.15) by ∂_m, (2.16) by ∂_j, subtracting and using (2.15), (2.16) again.

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The above discussion applied to the case of the H_3 background of the form given in (2.2). In general, one can show that there are no "supernumerary" supersymmetries for any pp-wave background supported by any *non-constant* H_3 form. Indeed, consider the most general ansatz for the "null' 2-form field: $B_2 = b_s(x)du \wedge dx^s$. Here x^s are all 8 transverse coordinates, i.e. this includes (2.2) as a special case. Then $H_{usr} \equiv f_{sr} = \partial_s b_r - \partial_r b_s$, and thus the s-component of the gravitino equation gives (cf. (2.15), (2.16))

$$\partial_s \epsilon + \frac{1}{4} f_{sr}(x) \Gamma^{ur} \epsilon = 0 \,.$$

Let us assume that, e.g., $f_{12} \neq 0$. Acting on the $s = 1$ equation by ∂_2 and on the $s = 2$ equation by ∂_1, and subtracting, we get $\partial_s f_{12} \Gamma^{us} \epsilon = 0$. For non-constant f_{12}, this can be satisfied only if $\Gamma^u \epsilon = 0$. Again, this implies [6, 32, 10] that the light-cone GS lagrangians corresponding pp-wave backgrounds of the form $ds^2 = dudv + K(x)du^2 + dx_s^2$ supported by non-constant H_{usr} or F_{usr} forms will not have linearly-realized world-sheet supersymmetry.

The case (2.5) of constant H_3 (i.e. $\partial_i \partial_j b_m = 0$) is special. Here the gravitino condition does not reduce the number of supersymmetries. For generic constant H_3 configuration in type IIB theory the dilatino variation equation reduces the number of unbroken supersymmetries to 16, as one is to impose $\Gamma^u \epsilon = 0$ to satisfy (2.13). However, for special "self-dual" matrices $\partial_i b_m$ (which correspond, in particular, to the case of the direct sum of the two Nappi-Witten [11] models which is the Penrose limit of the $AdS_3 \times S_3$ background) the condition (2.13) breaks less than 16 supersymmetries.[12]

3. pp-wave backgrounds parametrized by holomorphic functions

Here we shall specialize to a particular subset of the above backgrounds which may be viewed as NS-NS analogs of the R-R 5-form backgrounds discussed in [10]. They can be found from the general exact solution (2.1)–(2.4) with the even number $d = 2n$ of x_i coordinates and thus $8 - 2n$ "spectator" coordinates y_m organized into two sets of n and $4 - n$ complex coordinates z_a and z_α, respectively. The Laplace equation for b_m in (2.3) can then be solved in terms of holomorphic and anti-holomorphic functions. The general structure of the solution written in complex coordinates is then

$$ds^2 = dudv + K(z_a, z_a^*)du^2 + dz_a dz_a^* + dz_\alpha dz_\alpha^* \,, \tag{3.1}$$

$$B_2 = \frac{1}{2}\eta_\alpha(z_b) du \wedge dz_\alpha + \text{c.c.}\,, \quad H_3 = -\frac{1}{2} du \wedge \omega_2 \,, \quad \omega_2 = \partial_a \eta_\alpha(z_b) dz_a \wedge dz_\alpha + \text{c.c.}\,, \tag{3.2}$$

$$\eta_\alpha = \eta_\alpha(z_1, \ldots, z_n) \,, \qquad a = 1, \ldots, n \,, \quad \alpha = n+1, \ldots, 4 \,. \tag{3.3}$$

The cases of $n = 1, 2, 3$ represent inequivalent solutions. A more general solution (belonging again to the family of solutions of section 2.1) can be obtained by choosing[13]

$$\omega_2 = \partial_a \eta_\alpha(z_b) dz_a \wedge dz_\alpha + \partial_a \rho_\alpha(z_b) dz_a \wedge dz_\alpha^* + \text{c.c.}\,. \tag{3.4}$$

[12] In the example of the Penrose limit of $AdS_3 \times S_3$, the resulting number of unbroken supersymmetries, i.e. the solutions to $\Gamma^u(1 - \Gamma_{1234})\epsilon = 0$, is 24.

[13] The particular case $\eta_\alpha = 0$ is equivalent to the solution (3.2) by the simple change of coordinate $z_\alpha \to z_\alpha^*$.

3.1 Particular examples

Simple examples of solutions are found by choosing:

$$n = 1: \quad \omega_2 = \partial_1\eta(z_1)dz_1 \wedge dz_3 + \text{c.c.} \tag{3.5}$$

$$n = 2: \quad \omega_2 = \partial_1\eta(z_1)dz_1 \wedge dz_3 + \partial_2\tilde{\eta}(z_2)dz_2 \wedge dz_4 + \text{c.c.} \tag{3.6}$$

$$n = 3: \quad \omega_2 = \partial_1\eta(z_1, z_2)dz_1 \wedge dz_4 + \partial_2\eta(z_1, z_2)dz_2 \wedge dz_4 + \text{c.c.} \tag{3.7}$$

Eq. (3.5) is a special case of (3.2) with $\eta_3 = \eta(z_1)$ and $\eta_2 = \eta_4 = 0$. The $n = 2$ example (3.6) is obtained from (3.2) by setting $\eta_3 = \eta(z_1)$ and $\eta_4 = \tilde{\eta}(z_2)$. The more general $n = 3$ case has $\eta = \eta_4(z_1, z_2, z_3)$ (then ω_2 has one extra term containing dz_3).

For the examples in (3.5) and (3.6), eq. (2.3) has the following special solutions for K (we choose $K_0 = 0$ in (2.4))

$$n = 1: \quad K = -\frac{1}{2}|\eta(z_1)|^2 \,, \tag{3.8}$$

$$n = 2: \quad K = -\frac{1}{2}\left(|\eta(z_1)|^2 + |\tilde{\eta}(z_2)|^2\right) . \tag{3.9}$$

It is useful also to record the form of H_3 in (3.5) and (3.6) in real coordinates. Defining

$$z_1 = x_1 + ix_3 \,, \qquad z_2 = x_5 + ix_7 \,, \qquad z_3 = x_2 + ix_4 \,, \qquad z_4 = x_6 + ix_8 \,,$$

we get

$$\begin{aligned} n = 1: \quad \omega_2 =& (\partial_1\eta + \partial_1^*\eta^*)(dx_1 \wedge dx_2 - dx_3 \wedge dx_4) + i(\partial_1\eta - \partial_1^*\eta^*) \times \\ & \times (dx_1 \wedge dx_4 + dx_3 \wedge dx_2) \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} n = 2: \quad \omega_2 =& (\partial_1\eta + \partial_1^*\eta^*)(dx_1 \wedge dx_2 - dx_3 \wedge dx_4) + i(\partial_1\eta - \partial_1^*\eta^*) \times \\ & \times (dx_1 \wedge dx_4 + dx_3 \wedge dx_2) + (\partial_2\tilde{\eta} + \partial_2^*\tilde{\eta}^*)(dx_5 \wedge dx_6 - dx_7 \wedge dx_8) + \\ & + i(\partial_2\tilde{\eta} - \partial_2^*\tilde{\eta}^*)(dx_5 \wedge dx_8 + dx_7 \wedge dx_6) . \end{aligned} \tag{3.11}$$

Here for convenience we are using the same notation x_i for all of the transverse coordinates, instead of the splitting them into (x_i, y_m) as we did in section 2.

It is easy to read off the relation between η_α in (3.2) and b_m appearing in the general expression (2.2). For example, in the $n = 1$ case (3.10), x_2, x_4 play the role of the two "free" y_m coordinates and we have two components of b_m which depend on "dynamical" coordinates x_1, x_3

$$b_2 = b_2(x_1, x_3) = \text{Re } \eta \,, \qquad b_4 = b_4(x_1, x_3) = -\text{ Im } \eta \,,$$

so that

$$\partial_1 b_2 = -\partial_3 b_4 = \text{Re } (\partial_1\eta) = \frac{1}{2}(\partial_1\eta + \partial_1^*\eta^*) \,, \qquad \partial_3 b_2 = \partial_1 b_4 = -\text{Im } (\partial_1\eta) = \frac{1}{2}i(\partial_1\eta - \partial_1^*\eta^*) . \tag{3.12}$$

3.2 Space-time supersymmetry

Let us now count the number of unbroken supersymmetries for these solutions, solving the conditions (2.17), (2.19) in the above special cases. First, the condition (2.17) breaks 16 supersymmetries. Consider now the remaining equation (2.19) . In the $n = 1$ case, using (3.10), we get the restriction[14]

$$(1 + \Gamma_{1234})\epsilon = 0 \tag{3.13}$$

which breaks 8 more supersymmetries, so that there are 8 remaining supersymmetries. In the $n = 2$ case, using (3.11), we obtain

$$(1 + \Gamma_{1234})\epsilon = 0\,, \qquad (1 + \Gamma_{5678})\epsilon = 0\,, \tag{3.14}$$

and, as a result, there are 4 unbroken supersymmetries.

In a generic $n = 1$ model, one has

$$\omega_2 = \partial_1\eta_\alpha(z_1)dz_1 \wedge dz_\alpha + \text{c.c.}\,, \tag{3.15}$$

where η_α ($\alpha = 2, 3, 4$) are 3 independent functions. Here the condition (2.19) becomes

$$(1 + \Gamma_{1234})\epsilon = 0\,, \qquad (1 + \Gamma_{1537})\epsilon = 0\,, \qquad (1 + \Gamma_{1638})\epsilon = 0\,. \tag{3.16}$$

This leads to two unbroken supersymmetries. The same conclusion is reached for a generic $n = 3$ model with $\omega_2 = \partial_a\eta_4(z_1, z_2, z_3)dz_a \wedge dz_4 + \text{c.c.}$, where (2.19) leads again to three conditions of the form (3.16), i.e. to two unbroken supercharges.

The generic $n = 2$ case

$$\begin{aligned}\omega_2 = {} & \partial_1\eta_3(z_1, z_2)dz_1 \wedge dz_3 + \partial_2\eta_3(z_1, z_2)dz_2 \wedge dz_3 + \\ & + \partial_1\eta_4(z_1, z_2)dz_1 \wedge dz_4 + \partial_2\eta_4(z_1, z_2)dz_2 \wedge dz_4 + \text{c.c.}\end{aligned} \tag{3.17}$$

requires a closer examination. By writing (3.17) in cartesian coordinates, we find that for arbitrary functions $\eta_3(z_1, z_2), \eta_4(z_1, z_2)$, the supersymmetry condition (2.19) gives

$$(1+\Gamma_{1234})\epsilon = 0\,, \qquad (1+\Gamma_{5274})\epsilon = 0\,, \qquad (1+\Gamma_{5678})\epsilon = 0\,, \qquad (1+\Gamma_{1638})\epsilon = 0\,. \tag{3.18}$$

Since the last condition follows from the first three, we conclude that the generic $n = 2$ case also preserves two supersymmetries.

4. String sigma model actions for the NS-NS pp-wave backgrounds

4.1 Covariant action

The bosonic part of the sigma model lagrangian corresponding to the generic background (2.1), (2.2), (2.4) (with $K_0 = 0$) is given by[15]

$$\begin{aligned}L_B = {} & \partial_+u\partial_-v - \frac{1}{2}b_m^2(x)\partial_+u\partial_-u + b_m(x)(\partial_+u\partial_-y_m - \partial_+y_m\partial_-u) + \\ & + \partial_+x_i\partial_-x_i + \partial_+y_m\partial_-y_m\,.\end{aligned} \tag{4.1}$$

[14]From (3.10) we find that (2.19) takes the form $[a(x)(\Gamma_{12} - \Gamma_{34}) + b(x)(\Gamma_{14} - \Gamma_{23})]\epsilon = 0$ which can be satisfied for any functions a, b if we require $(\Gamma_{12} - \Gamma_{34})\epsilon = 0$ and $(\Gamma_{14} - \Gamma_{23})\epsilon = 0$. Each of these two conditions is equivalent to $(1 + \Gamma_{1234})\epsilon = 0$.

[15]We shall use Minkowski world-sheet coordinates with $\sigma^\pm = \tau \pm \sigma$, and $\partial_\pm = (1/2)(\partial_\tau \pm \partial_\sigma)$. The string action is $S = 1/(\pi\alpha')\int d^2\sigma\, L$. The space-time light-cone coordinates are $u = y - t,\ v = y + t$.

Note that by applying 2-d duality (i.e. T-duality in the target space) in y_m we get a model with zero 3-form field but off-diagonal metric, i.e. (4.1) becomes

$$\tilde{L}_B = \partial_+ u \partial_- v - \frac{1}{2} b_m^2(x) \partial_+ u \partial_- u + b_m(x)(\partial_+ u \partial_- \tilde{y}_m + \partial_+ \tilde{y}_m \partial_- u) + \\ + \partial_+ x_i \partial_- x_i + \partial_+ \tilde{y}_m \partial_- \tilde{y}_m \,. \tag{4.2}$$

In general, the fermionic part of the (1,1) world-sheet supersymmetric sigma model can be written in terms of the generalized Lorentz connections $\omega^{\hat{\rho}}_{\pm\hat{\nu}\mu} = \omega^{\hat{\rho}}_{\hat{\nu}\mu} \pm (1/2) H^{\hat{\rho}}{}_{\hat{\nu}\mu}$

$$L_F = i\lambda_{R\hat{\rho}}(\delta^{\hat{\rho}}_{\hat{\nu}}\partial_+ + \omega^{\hat{\rho}}_{-\hat{\nu}\mu}\partial_+ x^\mu)\lambda^{\hat{\nu}}_R + i\lambda_{L\hat{\rho}}(\delta^{\hat{\rho}}_{\hat{\nu}}\partial_- + \omega^{\hat{\rho}}_{+\hat{\nu}\mu}\partial_- x^\mu)\lambda^{\hat{\nu}}_R + \frac{1}{2}\hat{R}_{+\hat{\rho}\hat{\nu}\hat{\sigma}\hat{\mu}}\lambda^{\hat{\rho}}_L\lambda^{\hat{\nu}}_L\lambda^{\hat{\sigma}}_R\lambda^{\hat{\mu}}_R \,. \tag{4.3}$$

In the present case of (2.1), (2.2), (2.4)[16]

$$\omega_{\pm\hat{u}\hat{m}} = \pm\frac{1}{2}\partial_i b_m dx_i \,, \qquad \omega_{\pm\hat{i}\hat{m}} = \mp\frac{1}{2}\partial_i b_m \, du \,, \qquad \omega_{\pm\hat{u}\hat{i}} = \frac{1}{2}\partial_i K du \mp \frac{1}{2}\partial_i b_m dy_m \,. \tag{4.4}$$

and the non-zero components of $\hat{R}_{+uiuj} = \hat{R}_{-uiuj}$ were given in (2.12).

The explicit form of the lagrangian (4.2) in the case of the $n = 1$ solution (3.1), (3.2) and (3.5) parametrized by an arbitrary holomorphic function $\eta = \eta(z_1)$ is ($\alpha = 2, 3, 4$)

$$L_B = \partial_+ u \partial_- v - \frac{1}{2}\eta\eta^* \partial_+ u \partial_- u + \frac{1}{2}\eta(\partial_+ u \partial_- z_3 - \partial_+ z_3 \partial_- u) + \frac{1}{2}\eta^*(\partial_+ u \partial_- z_3^* - \partial_+ z_3^* \partial_- u) + \\ + \partial_+ z_1 \partial_- z_1^* + \partial_+ z_\alpha \partial_- z_\alpha^* \,. \tag{4.5}$$

Similarly, by applying T-duality transformations in $z_3 = x_2 + ix_4$ we get a special case of pure-metric model (4.2)

$$\tilde{L}_B = \partial_+ u \partial_- v - \frac{1}{2}\eta\eta^* \partial_+ u \partial_- u + \frac{1}{2}\eta(\partial_+ u \partial_- z_3 + \partial_+ z_3 \partial_- u) + \frac{1}{2}\eta^*(\partial_+ u \partial_- z_3^* + \partial_+ z_3^* \partial_- u) + \\ + \partial_+ z_1 \partial_- z_1^* + \partial_+ z_\alpha \partial_- z_\alpha^* \,. \tag{4.6}$$

4.2 Light-cone gauge action

Let us now consider the form of the string lagrangian in the light-cone gauge. Since u and $\lambda^u_{L,R}$ obey free field equations, we can supplement the superconformal gauge with the standard light-cone gauge conditions

$$u = 2\alpha' p^u \tau \,, \qquad \lambda^u_{L,R} = 0 \,. \tag{4.7}$$

Then the bosonic part of the lagrangian (4.2) takes the form

$$L_B = \partial_+ x_i \partial_- x_i - \frac{1}{2}\mathrm{m}^2 b_m^2(x) - \mathrm{m} b_m(x)(\partial_+ y_m - \partial_- y_m) + \partial_+ y_m \partial_- y_m \,, \\ \mathrm{m} \equiv \alpha' p^u = \partial_\pm u \,. \tag{4.8}$$

Note that in our notation $u, v, x^i, \sqrt{\alpha'}, (p^u)^{-1}$ and thus m have dimension of length, while the world-sheet coordinates τ and $\sigma \in [0, 2\pi)$ are dimensionless.[17] The components of

[16] Here $\eta_{\hat{u}\hat{v}} = 1/2$, $\eta^{\hat{u}\hat{v}} = 2$, etc. and hats on u, i, m indices can be omitted.

[17] The standard alternative is to rescale τ and σ by $\alpha' p^u$ (which is a symmetry of the $2 + d$ dimensional conformal theory), thus giving them dimension of length. Then in the light-cone gauge $x^+ = 2\tau$, $0 \leq \sigma < 2\pi\alpha' p^u$.

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the metric and 2-form tensor, i.e. K and b_m in (2.1), (2.2) or $\eta_\alpha(z)$ in (3.2), are also dimensionless. From the world-sheet point of view it is more natural to treat u, v, x^i as dimensionless while τ, σ as having dimension of length. Then m has world-sheet dimension of mass.

In view of (2.3), (2.12), the fermionic part of the action (4.3) becomes quadratic in fermions

$$L_F = i\left[\lambda_R^i \partial_+ \lambda_R^i + \lambda_L^i \partial_- \lambda_L^i + \mathrm{m}\partial_i b_m(x)(\lambda_R^i \lambda_R^m - \lambda_L^i \lambda_L^m) + \lambda_R^m \partial_+ \lambda_R^m + \lambda_L^m \partial_- \lambda_L^m\right]. \quad (4.9)$$

The sigma model action (4.8), (4.9) follows also directly from the (1,1) superfield form of the action $\int d^2\sigma d^2\vartheta (G_{\mu\nu} + B_{\mu\nu})(\hat{X}) D_+ \hat{X}^\mu D_- \hat{X}^\nu$ with $B_{um} = b_m$ and the superfield $\hat{X}^u$ chosen in the light-cone gauge form $\hat{X}^u = u = 2m\tau$. The latter choice breaks the 2-d Lorentz invariance and the (1,1) 2-d supersymmetry.

The resulting "transverse" gauge-fixed 1+1 dimensional theory is thus not 2-d Lorentz-covariant — the bosonic term originating from the B_2-coupling contains explicit sigma-derivative (or $\partial_+ - \partial_-$) term. The absence of manifest 2-d Lorentz covariance of the "transverse" theory is not unfamiliar: it is generic to many similar pp-wave string models written in the light-cone gauge. In particular, the Lorentz covariance is absent in all cases where K has u- (and thus τ-) dependence.

The lagrangian (4.8) and (4.9) may be interpreted as describing a system of chiral scalars $y^m_{L,R}$ and their (1,0) and (0,1) superpartners $\lambda^m_{L,R}$ interacting with scalars x^i and fermions $\lambda^i_{L,R}$. There is an obvious left-right decomposition in the y^m, λ^m sector, but the two chiral sectors are mixed by the interaction terms in the x^i, λ^i sector.

5. Light-cone gauge GS string actions for the non-constant R-R pp-wave backgrounds

By applying S-duality transformation to the background (2.1), (2.2) and (2.4), i.e. replacing H_3 by F_3, we obtain another solution of type-IIB supergravity theory which has constant dilaton and non-trivial R-R 3-form field. According to the argument in section 2.2, this R-R background, just like its NS-NS counterpart, is, in fact, an exact solution of type IIB string theory. The conditions of space-time supersymmetry are again determined by (2.17) and (2.19) .

To find the form of the corresponding light-cone gauge Green-Schwarz action we follow the same logic as was used in [7, 8]. As was mentioned in [7, 8] in the case of constant R-R p-form strengths, and as we shall explicitly prove below in the general case of non-constant R-R pp-wave backgrounds, the property that the curvature and F_p have null structure implies that all higher than quadratic terms in fermions should be absent from the GS action written in the light-cone gauge

$$\Gamma^u \theta^{1,2} = 0, \qquad u = 2\mathrm{m}\tau, \qquad \mathrm{m} \equiv \alpha' p^u, \quad (5.1)$$

while the quadratic fermionic term is essentially determined [7, 8] (see also [36, 35]) by the structure of the corresponding generalized covariant derivative[18]

$$L_{2F} = i\partial_- X^\mu \ \bar\theta^1 \Gamma_\mu \hat D_+ \theta^1 + i\partial_+ X^\mu \ \bar\theta^2 \Gamma_\mu \hat D_- \theta^2 \,, \tag{5.2}$$

where in the F_3-case

$$\hat D_\mu \theta^{1,2} = \left(\partial_\mu + \frac{1}{4}\omega_{\sigma\lambda\mu}\Gamma^{\sigma\lambda}\right)\theta^{1,2} - \frac{1}{8\cdot 3!}F_{\sigma\nu\lambda}\Gamma^{\sigma\nu\lambda}\Gamma_\mu\theta^{2,1} \,. \tag{5.3}$$

5.1 Light-cone gauge GS action is quadratic in fermions for generic pp-wave background

The form of the covariant GS action in a generic type II supergravity background is very complicated, containing terms of all possible powers in the two 10-d MW spinor variables θ^I which come out of the component expansions of the superfields entering the superspace form of the action [29]. For certain backgrounds, the action written in a special κ-symmetry gauge may become quartic in fermions (as is the case for the $AdS_5\times S^5$ action in the light-cone gauge [30], see also [34]). A particularly simple case is that of the homogeneous plane-wave backgrounds [9] — here the light-cone gauge GS action turns out to be quadratic in fermions [5].

Below we shall prove that this is true also for generic inhomogeneous ($R\neq$ const, $F_p\neq$ const) pp-wave backgrounds with the metric admitting a covariantly constant null Killing vector (i.e., for example, (2.1)) and the R-R or NS-NS p-form strengths having the "null" structure (2.7). The key property we will use is that the curvature and the p-form strengths do not have lower v-components and do not depend on v. Another important point is that the curved-space GS action (which is a "supersymmetrization" of the standard bosonic sigma model action) is quadratic in 2-d derivatives ∂_a. This implies that generic fermionic terms can only be of the following 3 types

$$\begin{aligned} L_1 &= D^nR\ldots D^kF\ldots\bar\theta\ldots\theta\ldots\bar\theta\ldots\theta\ \bar\theta\ldots\partial\theta\ \partial X \,,\\ L_2 &= D^nR\ldots D^kF\ldots\bar\theta\ldots\theta\ldots\bar\theta\ldots\theta\ \partial X\partial X \,,\\ L_3 &= D^nR\ldots D^kF\ldots\bar\theta\ldots\theta\ldots\bar\theta\ldots\theta\ \bar\theta\ldots\partial\theta\ \bar\theta\ldots\partial\theta \,, \end{aligned} \tag{5.4}$$

where X are the bosonic coordinates $X^\mu=(u,v,x^i)$, and dots between $\bar\theta$ and θ stand for products of Γ-matrices.

To get a non-zero result from a particular term after imposing the light-cone gauge condition $\Gamma^u\theta=0$ each $\bar\theta\ldots\theta$ factor in (5.4) must have the form $\bar\theta\Gamma^v\Gamma^{i_1}\ldots\Gamma^{i_n}\theta$. Since the background-dependent factor $D^nR\ldots D^kF$ cannot have lower v-indices, each Γ^v must be accompanied by $\partial X^u\equiv\partial u$. That immediately implies that all L_3-terms in (5.4) must vanish, while L_1 must be quadratic and L_2 — at most quartic in θ. In addition, all non-trivial $D^nR\ldots D^kF$ factors must contain at least one lower u-index which is to be

[18] Here we use the standard normalization in which $e^\phi=1$, so that the R-R terms in the covariant derivative in [8, eq. (5.4)] should be multipled by extra 1/2.

contracted with either ∂X^u or some Γ^u in the fermionic factor; in the latter case, such term vanishes.[19]

This implies the vanishing of all "non-flat" terms in L_1 and quartic fermionic terms in L_2, and thus leaves us with the standard flat-space GS term $L_1 = \partial u\bar{\theta}\Gamma^v\partial\theta$ as well as with the following candidates for the non-vanishing quadratic fermionic terms

$$L_2 = D_{i_1} \dots D_{i_m} F_{uj_1\dots j_{p-1}}\ \partial u\ \bar{\theta}\Gamma^v\Gamma^{i_1}\dots\Gamma^{j_{p-1}}\theta\,. \tag{5.5}$$

Furthermore, the only term of that type that can actually appear in the GS action should contain *no* covariant derivatives acting on F_p — each covariant derivative would be accompanied by an extra $\bar{\theta}\dots\theta$ factor. Indeed, this follows simply from dimensional considerations: if X has dimension of length l (e.g., $\sqrt{\alpha'}$), then θ has dimension $l^{1/2}$, the metric and $(p-1)$-form potentials are dimensionless, and thus D_μ and $F_{\mu_1\dots\mu_p}$ have dimensions l^{-1} (while $R_{\mu\nu\lambda\rho}$ — dimension l^{-2}).

As a result, we arrive at the same form of the action as in the case of the constant null R-R flux [5, 7, 8]: the only non-trivial coupling to the background field strength is through the generalized covariant derivative that enters the gravitino supersymmetry transformation rule, i.e. (5.3) (with other p-form terms and the dilaton e^ϕ factors included in general [31]).

To summarize, the above argument allows one to determine the form of the light-cone gauge GS action for any inhomogeneous R-R pp-wave background, in particular, for the F_5-form background in [10]. The general structure of the GS action corresponding to the metric (2.1) supported by a R-R background $F_p(x)$ of the form (2.7), which solves the supergravity equations of motion (i.e. $R_{uu} \sim F_{u\dots}F_{u\dots}$, $\partial_i F^{iu\cdots} = 0$), written in the light-cone gauge (5.1) is thus (ignoring explicit value of the numerical normalization factor c_p in the last term)

$$L = \partial_+ x_i \partial_- x_i + \mathrm{m}^2 K(x) + i\theta^1\Gamma^v\partial_-\theta^1 + i\theta^2\Gamma^v\partial_+\theta^2 + i\mathrm{m}c_p F_{ui_1\dots i_{p-1}}(x)\theta^1\Gamma^v\Gamma^{i_1\dots i_{p-1}}\theta^2\,. \tag{5.6}$$

5.2 UV finiteness of the light-cone GS theory

Viewing (5.6) as a 2-d field theory, it is natural to assign 2-d length dimensions to τ and σ and thus to assume that other dimensions are $[x] = 0$, $[\theta] = -1/2$, $[\mathrm{m}] = -1$. Then $K(x)$ and $F_p(x)$ are dimensionless, and dimensional analysis implies that the only possible logarithmically[20] divergent l-loop counterterms in this theory should be proportional to m^2. Therefore, they must be linear in derivatives of K and quadratic in derivatives of F_p, i.e.

$$\Delta L^{(l)} = \mathrm{m}^2 \left[a_l\ \partial^{2l} K(x) + b_l\ \partial^{l-1} F_{ui_1\dots i_{p-1}}(x)\ \partial^{l-1} F_{ui_1\dots i_{p-1}}(x)\right]\,, \quad l = 1, 2, \dots\,, \tag{5.7}$$

where ∂^{l-1} stands for $\partial_{j_1}\cdots\partial_{j_{l-1}}$, etc. These are the same counterterms as expected (2.8), (2.10) on the general grounds in the covariant theory before the light-cone gauge fixing

[19] More precisely, ∂X^u may be also contracted with Γ_u between $\bar{\theta}$ and θ but, since $\Gamma^u\Gamma_u = 2$ when acting on fermions subject to the light-cone gauge condition, this leads to an equivalent conclusion.

[20] Trivial quadratic divergences cancel because of equal total numbers of bosons and fermions.

(note that here $R_{uiuj} = -(1/2)\partial_i\partial_j K$ and $\mathrm{m} = \partial_\pm u$). The coefficients in (5.6) are such that the 1-loop divergences cancel (due to the relation between K and F_p as in (2.3), (2.4)). Denoting by $G(\xi-\xi')$ the propagator of the 2-d bosons $x_i(\xi)$ ($\xi^a = (\tau,\sigma)$), the propagators of the fermions θ_L and θ_R are then $\partial_+ G(\xi-\xi')$ and $\partial_- G(\xi-\xi')$, and thus the coefficients in (5.7) are

$$a_l \sim \int d^2\xi\ [G(\xi-\xi')]^l_{\xi\to\xi'}\,, \qquad b_l \sim \int d^2\xi d^2\xi'\ [G(\xi-\xi')]^{l-1}\partial_+ G(\xi-\xi')\partial_- G(\xi-\xi')\,.$$

Integrating by parts in the expression for b_l we can transform it to the same form as a_l: $b_l \sim \int d^2\xi d^2\xi'\ [G(\xi-\xi')]^l \partial_+\partial_- G(\xi-\xi') \sim \int d^2\xi\ [G(\xi-\xi')]^l_{\xi\to\xi'}$. Since $\partial_{i_1} F_{ui_1\ldots i_{p-1}} = 0$, $\partial^2 F_{ui_1\ldots i_{p-1}} = 0$ the cancellation seen at 1-loop order should continue at higher loops.[21]

This conclusion is of course in agreement with the general finiteness argument given in section 2.2. As a result, the light-cone gauge theory is UV finite (but its scale invariance explicitly broken by the presence of the "mass" parameter m).

5.3 Explicit form of the light-cone GS action for the R-R 3-form pp-wave background

Let us now return to the specific case of our interest, namely, the background (2.1), (2.2) with H_3 replaced by F_3. As in [8], we shall keep the free-theory notation $\theta^1 \equiv \theta_L$, $\theta^2 \equiv \theta_R$. Then the corresponding light-cone gauge GS lagrangian is given by the sum of the following bosonic and fermionic parts (cf. (4.8), (4.9), (5.2))[22]

$$\begin{aligned} L_B &= \partial_+ x_i \partial_- x_i - \frac{1}{2}\mathrm{m}^2 b_m^2(x) + \partial_+ y_m \partial_- y_m\,, \\ L_F &= i\theta_R\gamma^v\partial_+\theta_R + i\theta_L\gamma^v\partial_-\theta_L - \frac{1}{4}i\mathrm{m}\partial_j b_m(x)\ \theta_L\gamma^v\gamma^{jm}\theta_R\,. \end{aligned} \tag{5.8}$$

As usual, we rescaled the fermions by power of p^u. For comparison, the fermionic term in the light-cone GS lagrangian for the NS-NS background (2.2) is (cf. its RNS form in (4.9))

$$L_F = i\theta_R\gamma^v\partial_+\theta_R + i\theta_L\gamma^v\partial_-\theta_L - \frac{1}{4}i\mathrm{m}\partial_j b_m(x)\ (\theta_R\gamma^v\gamma^{jm}\theta_R - \theta_L\gamma^v\gamma^{jm}\theta_L)\,. \tag{5.9}$$

Here b_m is any harmonic function. The background has residual space-time supersymmetry (so that (5.8) has global fermionic symmetry $\theta \to \theta + \epsilon$) provided b_m is such that eq. (2.19) has non-trivial solutions. As we have mentioned in section 2.3, the absence of supersymmetries with $\Gamma^u\epsilon \neq 0$ for $\partial_i\partial_j b_m \neq 0$ implies that in contrast to the case of constant F_3 (i.e. linear b_m [8]) background and the non-constant F_5 case in [10], here the light-cone gauge GS model (5.8) and (5.8) will not have an additional linearly realized 2-d supersymmetry.

[21]In a particular (e.g. dimensional) regularization scheme all higher tadpoles a_l, $l > 2$, can be set equal to zero (they lead only to higher than first powers of logarithm of the UV cutoff and thus can be ignored). In the case when the theory has extended 2-d supersymmetry the cancellation of the two contributions in (5.7) can be understood as a consequence of non-renormalization of the chiral superpotential. This is what happens, e.g., in the (2,2) supersymmetric model of [10] where the bosonic potential is $|\partial_i W|^2$ and the Yukawa coupling matrix is $\partial_i\partial_j W$.

[22]We follow the spinor notation of [8], i.e. switch to the 16-component notation for the spinors, with γ^μ being 16-component real and symmetric matrices which replace 32-component matrices Γ^μ.

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Let us specify now to the case of the R-R background parametrized by an arbitrary holomorphic function, i.e. to the "S-dual" of the $n = 1$ background (3.1)–(3.5). According to the discussion in section 3.2, this background, like its NS-NS counterpart, should be preserving 8 supersymmetries. Written in complex notation, the corresponding light-cone GS lagrangian takes the following form (cf. (4.5), (5.8), (3.12); here we use the notation $z_1 \equiv z$, $\partial_z \eta \equiv \eta'$)

$$L_B = \partial_+ z \partial_- z - \frac{1}{2}\mathrm{m}^2 |\eta(z)|^2 + \partial_+ z_\alpha \partial_- z^*_\alpha \,, \tag{5.10}$$

$$L_F = i\theta_R \gamma^v \partial_+ \theta_R + i\theta_L \gamma^v \partial_- \theta_L - \frac{1}{2} i\mathrm{m}\ \mathrm{Re}(\eta')\ \theta_L \gamma^v \gamma^{12} \hat{P} \theta_R + \frac{1}{2} i\mathrm{m}\ \mathrm{Im}(\eta')\ \theta_L \gamma^v \gamma^{14} \hat{P} \theta_R \,. \tag{5.11}$$

We have used (3.10), and introduced

$$\hat{P} \equiv \frac{1}{2}(1 + \gamma^{1234}) \,,$$

which is the same projector as appearing in eq. (3.13). By squaring the Yukawa coupling matrix in (5.11) it is easy to check that the number of interacting real fermions is 4. This does not match the number (two) of interacting real scalars $z = x_1 + ix_3$ — as expected, the model we got can not be 2-d supersymmetric.

Explicitly, by choosing an appropriate representation of the γ-matrices (see, e.g. [8, section 5.2]), the light-cone gauge condition $\gamma^u \theta_{L,R} = 0$ can be solved in terms of $8 + 8$ independent real fermions $S_{L,R}$. Then $4 + 4$ of the fermions will be interacting and $4 + 4$ will be massless (the latter are the ones which are annihilated by $\hat{P}$). By further specifying the γ-matrix representation, the part of the lagrangian (5.10), (5.11) describing the system of interacting bosons and fermions can be written as

$$\begin{aligned} L_{\mathrm{int}} &= \partial_+ z \partial_- z^* - \frac{1}{2}\mathrm{m}^2 \eta(z)\eta^*(z^*) + i\psi^*_{kR}\partial_+\psi_{kR} + i\psi^*_{kL}\partial_-\psi_{kL} \\ &\quad + \frac{i}{2}\mathrm{m}\left[\eta'(z)\psi_{kL}\psi_{kR} + \eta'^*(z^*)\ \psi^*_{kL}\psi^*_{kR}\right] . \end{aligned} \tag{5.12}$$

Here $\psi_{1\ L,R}$ and $\psi_{2\ L,R}$ are two complex combinations of $4 + 4$ components of $S_{L,R}$, and the sum over $k = 1, 2$ is implied.[23]

This model is very similar to the (2,2) supersymmetric model $L_{(2,2)} = \int d^4\vartheta\ \Phi^*\Phi + \int d^2\vartheta\ W(\Phi) + \text{c.c.}$, where $\Phi = z + \vartheta_1\psi_L + \vartheta_2\psi_R + \cdots$ and the superpotential $W(z)$ is related to an arbitrary function $\eta(z)$ by $W' = \eta$. Indeed, written in components $L_{(2,2)}$ has the same bosonic part as (5.12) and similar Yukawa terms, but just one instead of two pair of complex fermions ψ_L, ψ_R, i.e.

$$\begin{aligned} L_{(2,2)} &= \partial_+ z \partial_- z^* - \frac{1}{2}\mathrm{m}^2 \eta(z)\eta^*(z^*) + i\psi^*_R\partial_+\psi_R + i\psi^*_L\partial_-\psi_L + \\ &\quad + \frac{i}{\sqrt{2}}\mathrm{m}\ \left[\eta'(z)\ \psi_L\psi_R + \eta'^*(z^*)\psi^*_L\psi^*_R\right] . \end{aligned} \tag{5.13}$$

[23]This can be shown, e.g., by using 4 complex combinations $\hat{\gamma}_i$ of γ-matrices as in [10] and writing $S = \psi_i \hat{\gamma}^i \epsilon_0 + \text{c.c.}$, where ϵ_0 is a constant spinor satisfying $\hat{\gamma}_i \epsilon_0 = 0$.

Remarkably, both theories (5.12) and (5.13) are UV finite: the mismatch in the number of interacting fermions is compensated by different coefficients in front of the Yukawa coupling terms.[24]

The (2,2) supersymmetric model (5.13) was found in [10] to be a special case of the light-cone GS action for a particular pp-wave background supported by a nonconstant F_5-field. The difference between (5.13) and our model (5.12) corresponding to the F_3-background is not unexpected, given that the GS fermionic couplings in (5.6) corresponding to the two different exact string solutions — a background with R-R F_3 strength and a background with R-R F_5 strength supporting the same pp-wave metric — have different Γ-matrix structure.

While the model (5.12) does not have 2-d supersymmetry,[25] the two models are very similar, representing two different UV finite fermionic extensions of the same interacting bosonic theory with potential $V = \mathrm{m}^2|\eta(z)|^2$. For the special choices of $\eta(z)$ for which the bosonic theory (5.10) and its 2-d supersymmetric version (5.13) are integrable, the same integrability property is likely to be shared also by the theory (5.12).

With this motivation in mind we discuss in the next section 6 two examples of the holomorphic functions $\eta(z)$ which correspond to integrable models. As a preparation for the study of string spectrum of these models we shall make some remarks on the solutions of the Laplace equation in the corresponding metrics.

6. Examples related to integrable models

Let us consider some examples of the R-R light-cone gauge models parametrized by holomorphic functions in the case of the $n = 1$ model (5.10), (5.11), where interaction terms depend on only one complex coordinate field $z_1 \equiv z$. An arbitrary holomorphic function $\eta(z)$ which enters (5.10), (5.11) or (5.12) can be chosen, as in [10], to represent an integrable 2-d theory. We shall discuss two examples of $\eta(z)$ for which the corresponding (2,2) supersymmetric theories (5.13) are the $N = 2$ super sine-Gordon model [37] and the $N = 2$ super Liouville model [38]. Since the bosonic parts of (5.13) and (5.12) are the same, these examples are of interest also in the non-supersymmetric case of the model (5.12).

[24]In fact, any model with one complex scalar field and K species of fermions (generalising the $K = 1$ (5.13) and $K = 2$ (5.12) models) with the Yukawa coupling $i/(\sqrt{2K})\mathrm{m}\,[\eta'(z)\psi_{kL}\psi_{kR} + \eta'^*(z^*)\ \psi^*_{kL}\psi^*_{kR}]$ $(k = 1, \ldots, K)$ is also UV finite. The cancellation of divergences can be readily checked by repeating the argument given in section 5.2 (note that only single fermionic loop may contribute to the divergences). As a quick check of numerical coefficients one may consider the case of $\eta(z) = z$ and explicitly integrate out ψ^*_L and ψ^*_R in the path integral, getting a second-derivative action for ψ_L and ψ_R, whose contribution cancels the bosonic action contribution.

[25]In particular, the vacuum energy on the cylinder does not vanish. In the 1-loop approximation: $E_1 = \sum_{n=1}^{\infty}\left(6n + 2\sqrt{n^2+M^2} - 4n - 4\sqrt{n^2+(1/2)M^2}\right)$, where $M^2 = (1/2)\mathrm{m}^2|\eta'(z_0)|^2$ (z_0 = const.). This is to be compared to $E_1 = \sum_{n=1}^{\infty}\left(6n + 2\sqrt{n^2+M^2} - 6n - 2\sqrt{n^2+M^2}\right) = 0$ in the supersymmetric model (5.13).

6.1 $N = 2$ super Sine-Gordon case

Choosing $\eta(z)$ as ($z = x_1 + ix_3$, ω=real)

$$\eta(z) = \sin \omega z\,, \qquad \text{i.e.} \quad \eta = \sin \omega x_1 \cosh \omega x_3 + i \cos \omega x_1 \, \sinh \omega x_3\,, \tag{6.1}$$

the bosonic part of the GS lagrangian (5.10), (5.12) becomes the same as the bosonic part of the $N = 2$ super sine-Gordon model (5.13).

To get some elementary information about states of this theory which may be useful for future studies, here we shall determine the form of the simplest massless scalar vertex operator (i.e. the effective masses of scalar states in the supergravity part of the string spectrum). The discussion below applies also to the case of the model of [10] corresponding to the same pp-wave metric (2.1) supported not by the F_3 but by an F_5 background.

Let us consider a scalar fluctuation Ψ belonging to the massless supergravity multiplet, which obeys the curved-space Klein-Gordon equation.[26] Using eq. (3.1), we get

$$\partial_\mu(\sqrt{G}G^{\mu\nu}\partial_\nu)\Psi = \left[4\partial_u\partial_v - 4K(z,z^*)\partial_v^2 + \partial_i^2\right]\Psi = 0\,. \tag{6.2}$$

For the $N = 2$ sine-Gordon model (6.1) K (which is also the bosonic potential in 2-d model) is given by

$$K = -\frac{1}{2}|\sin(\omega z)|^2 = -\frac{1}{4}\left[\cosh(2\omega x_3) - \cos(2\omega x_1)\right]. \tag{6.3}$$

The general solution of (6.2) can be obtained as a linear combination of the waves

$$\Psi = e^{ip_u u + ip_v v + ip_s x_s}\, f(x_1) g(x_3)\,, \tag{6.4}$$

where x_s stand for the remaining free coordinates, and f and g satisfy

$$f''(x_1) + \left[a + p_v^2 \cos(2\omega x_1)\right] f = 0\,, \tag{6.5}$$

$$g''(x_3) + \left[e - p_v^2 \cosh(2\omega x_3)\right] g = 0\,, \tag{6.6}$$

$$4p_u p_v + p_s^2 + a + e = 0\,. \tag{6.7}$$

Eq. (6.5) is the well-known Schrödinger equation for the quantum pendulum, and its general solution is expressed in terms of Mathieu functions. If x_1 is non-compact then the parameter a takes continuous values. When x_1 is compact and has period $x_1 = x_1 + 2\pi/\omega$, there is a discrete spectrum of "Kaluza-Klein" momentum modes. Since the vertex operator must be a single-valued function of x_1, the Mathieu functions which are solutions of (6.5) should be $2\pi/\omega$ periodic in x_1. This is the case for certain values of a — the Mathieu characteristic eigenvalues — which thus determine the momentum modes in this sector.

[26] In general, the scalar anomalous dimension operator ($D^2 + \cdots$) that appears in the equation for the vertex operator may receive α'-corrections (see, e.g., [13, 28] and refs. there). For example, for the "null" NS-NS background in question (2.1), (2.2) in the bosonic string case one would get corrections like $H^{\mu\alpha\beta}H^{\nu}{}_{\alpha\beta}D_\mu D_\nu = 2(\partial_i b_m)^2\partial_v^2$, etc. However, in the superstring case such corrections are likely to vanish in an appropriate scheme.

The second equation (6.6) for $g(x_3)$ describes bound states. Its general solution is also expressed in terms of Mathieu functions (the associated Mathieu functions of the first kind). It is easy to find the energy eigenvalues using the WKB approximation. The WKB formula gives

$$I \equiv 2\int_0^{x_0} dx \sqrt{e - p_v^2 \cosh(2\omega x)} = \left(n + \frac{1}{2}\right)\pi\,, \qquad x_0 \equiv \frac{1}{2\omega}\text{arccosh}\frac{e}{p_v^2}\,. \tag{6.8}$$

The integral is expressed in terms of an elliptic function,

$$I = -2i\frac{\sqrt{e-p_v^2}}{\omega}\mathcal{E}(i\omega x_0, -\mu)\,, \qquad \mu \equiv \frac{2p_v^2}{e-p_v^2}\,. \tag{6.9}$$

For $e \gg p_v^2$, we can approximate (6.9) by

$$I \approx \frac{\sqrt{e}}{w}\log\frac{e}{p_v^2} \approx \left(n + \frac{1}{2}\right)\pi\,. \tag{6.10}$$

This determines the eigenvalues $e = e_n$ for large n.

The light-cone energy, i.e. the value of the light-cone hamiltonian on the corresponding supergravity (i.e. "massless" string) states is then given by

$$H = -p_u = \frac{p_s^2}{4p_v} + \frac{1}{4p_v}\,(a + e_n)\,. \tag{6.11}$$

As was noted above, in the case of compact $x_1 = x_1 + \frac{2\pi}{\omega}$, the parameter a takes a discrete set of values $a = a_r$ for which the Mathieu equation (6.5) admits periodic solutions. The energy of the physical states in this sector is then given by (6.11) with $a \to a_r$, i.e. is parametrized in terms of the transverse momentum p_s, and two integer numbers r and n, i.e. $H = H(p_s, r, n)$.

6.2 $N = 2$ super Liouville case

The super Liouville model was studied in the past, e.g., in the context of non-critical string theory [26]. The pp-wave framework of [10] and the present paper allows one to embed a model with a Liouville potential into string theory as a light-cone gauge theory corresponding to an exact string solution with constant dilaton field.

The super Liouville model is obtained by choosing $\eta(z)$ as ($z = x_1 + ix_3,\ \beta$=real)

$$\eta(z) = e^{\beta z}\,, \qquad \text{i.e.} \quad \eta = e^{\beta x_1}\cos\beta x_3 + ie^{\beta x_1}\sin\beta x_3\,, \qquad K = -\frac{1}{2}e^{2\beta x_1}\,. \tag{6.12}$$

One can also get Toda-type potentials by using more general models of section 3 with suitably chosen holomorphic functions η_a.

In the case of the NS-NS background the bosonic part of the corresponding covariant lagrangian (4.5) takes the following explicit form

$$\begin{aligned} L = {} & \partial_+ u\partial_- v - \frac{1}{2}e^{2\beta x_1}\partial_+ u\partial_- u + \partial_+ x_1\partial_- x_1 + \partial_+ x_3\partial_- x_3 + \partial_+ x_s\partial_- x_s + \\ & + e^{\beta x_1}\cos\beta x_3(\partial_+ u\partial_- x_2 - \partial_+ x_2\partial_- u) - e^{\beta x_1}\sin\beta x_3(\partial_+ u\partial_- x_4 - \partial_+ x_4\partial_- u)\,, \end{aligned} \tag{6.13}$$

where x_s stand for the remaining free bosonic coordinates.

In the R-R case, the light-cone GS lagrangian is given by (5.10), (5.11) (or by (5.12))

$$L_B = \partial_+ x_1 \partial_- x_1 - \frac{1}{2}\mathrm{m}^2 e^{2\beta x_1} + \partial_+ x_3 \partial_- x_3 + \partial_+ x_s \partial_- x_s \,, \tag{6.14}$$

$$\begin{aligned} L_F &= i\theta_R \gamma^v \partial_+ \theta_R + i\theta_L \gamma^v \partial_- \theta_L - \frac{1}{2} i\mathrm{m} e^{\beta x_1} \cos\beta x_3 \; \theta_L \gamma^v \gamma^{12} \hat{P} \theta_R + \\ &\quad + \frac{1}{2} i\mathrm{m} e^{\beta x_1} \sin\beta x_3 \; \theta_L \gamma^v \gamma^{14} \hat{P} \theta_R \,. \end{aligned} \tag{6.15}$$

According to the discussion in section 5, the fermion couplings here are different (cf. (5.13) and (5.12)) from those of the $N = 2$ super Liouville model, which itself can be obtained from the pp-wave background with F_5-field of [10] .

As in the sine-Gordon case, one gets two different models, depending on whether x_1 is compact or non-compact. In the case of compact x_1, the semi-classical regime of the Liouville model corresponds to large radius, whereas the quantum regime corresponds to small radius. In the later regime, as was pointed out in [10] , a more convenient description is in terms of a mirror theory. In the case of the $N = 2$ sine-Gordon theory, the mirror is a deformed CP^1 model, and it was argued in [10] that the mirror background cannot be a solution of supergravity as massive string modes are apparently excited. In the case of the $N = 2$ Liouville theory the mirror was shown [27] to be equivalent to the $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ Kazama-Suzuki model, which is a supersymmetric generalization of the 2d black hole model. It may be that in the super Liouville case the identification of a string background corresponding to the mirror theory is more direct.

Let us now follow section 6.1 and consider the form of the massless scalar vertex operator in the $N = 2$ super Liouville theory case. The vertex operator is again given by eq. (6.4), now with $g(x_3) = e^{ip_3 x_3}$, and with $f(x_1)$ determined by the following differential equation

$$f'' + \left(\nu^2\beta^2 - 2p_v^2 e^{2\beta x_1}\right) f = 0 \,, \qquad \nu^2\beta^2 = -(p_s^2 + 4p_u p_v) \,. \tag{6.16}$$

The general normalisable solution is given in terms of the Bessel functions I_ν

$$f(x_1) = i\left[I_{-i\nu}(c\; e^{\beta x_1}) - I_{i\nu}(c\; e^{\beta x_1})\right] \,, \qquad c = \sqrt{2}\,\frac{p_v}{\beta} \,. \tag{6.17}$$

The parameter ν takes continuous real values and represents momentum of the incoming/outgoing wave at $x \to -\infty$. At $x \to \infty$, the wave is suppressed exponentially due to the Liouville potential. The light-cone energy of this state is thus

$$H = -p_u = \frac{1}{4p_v}(p_s^2 + \nu^2\beta^2) \,. \tag{6.18}$$

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